DEVELOPMENT OF MATHEMATICS IN INDIA

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Course: HS 636

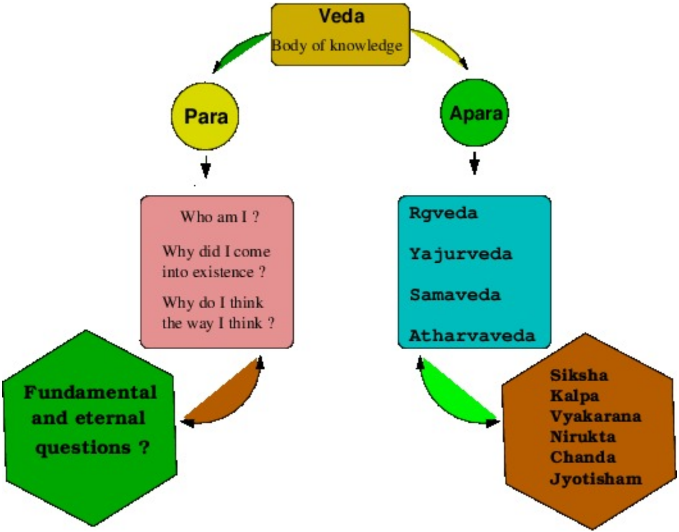
Vedas and Sulbasūtras - Part 1

Outline

- Introduction
- Mathematical references in Vedas
- What are ***Śulbasūtra*** texts?
- What does the word *Śulbasūtra* mean?
- Qualities of a *Śulbakāra*
- Finding the cardinal directions
- Methods for obtaining perpendicular bisector
- Bodhāyana method of constructing a square
- The *Śulba* theorem (Bodhāyana & Mānava version)
- A few triplets listed in *Śulbasūtras* (general principle?)
- Applications of *Śulba* theorem (next lecture?)

Introduction

Broad classification of Knowledge – *Muṇḍaka-Upaniṣad*



Introduction

Some Caveats and Observations (to avoid proliferation of false notions)

Statement of Sherlock Holmes, and an adapted version of it:[1]:

Anything which you may say will be used against you

Anything that you may say about Ancient India will be used against her.

A Catch-22, described by Bhartrhari:

मौनात् मूकः प्रवचनपटुः वातुलो जल्पको वा
धृष्टः पार्श्वे भवति च वसन् दूरतश्च अप्रगल्भः।
क्षान्त्या भीरुः यदि न सहते प्रायशो नाभिजातः
सेवाधर्मः परमगहनो योगिनामप्यगम्यः॥

If remains silent, he would be described as dumb, if speaks well as chatterer or prattler, would be damned impudent if moves very closely, and as a funk if he keeps distance; as timid if patient . . .

[1]Quoted by Amartya Dutta, ISI Calcutta

Introduction

Van der Waerden's observation on Brahmagupta's contribution

- The principle न कुर्यान्निष्फलम् कर्म is often quoted and generally upheld in Indian literature.
- Paraphrasing this, Saraswati Amma observes:

  > *Knowledge for its own sake did not appeal to Indian mind. Every discipline (śāstra) must have a purpose.*

- Taking this forward, the famous Dutch mathematician[2] (b.1903) Van der Waerden observes:

  > *Why should he (Brahmagupta) be interested in the solution of Pell's equation? . . . I can find only one explanation: he followed an earlier tradition derived from Greek sources ... . . I suppose the Greeks were able to solve Pell's equation, . . . I also suppose that their methods of calculation were copied, without proofs, in Hindu treatises like Brahmagupta's Siddhanta*

[2] who took Professorship at the age of 25.

Mathematical references in Vedas

Citations that unambiguously point to the decimal system being in vogue

- In *Kṛṣṇa-Yajur-Veda* we find an interesting passage wherein a sequence of ascending numbers appear in the context of offering venerations to Agni (Fire God).

सकृत्ते अग्ने नमः। द्विस्ते नमः। त्रिस्ते नमः।...

दशकृत्वस्ते नमः। शतकृत्वस्ते नमः। आसहस्रकृत्वस्ते नमः।

अपरिमितकृत्वस्ते नमः।

नमस्ते अस्तु मा मा हिꣳसीः।[3]

O fire, salutations unto you once. Salutations twice.
Salutations thrice ...
Salutations ten times. Salutations hundred times.
Salutations a thousand times.
Salutations unto you unlimited times.
My venerations to you, never ever hurt me.

[3] *Taittirīya Āraṇyakam* 4.69.

Mathematical references in Vedas

Citations that unambiguously point to the decimal system being in vogue

- We find yet another passage presenting a list of powers of 10 starting from hundred (10^2) to a trillion (10^{12}).

  शताय स्वाहा सहस्राय स्वाहायुताय स्वाहा नियुताय स्वाहा
  प्रयुताय स्वाहार्बुदाय स्वाहा न्यर्बुदाय स्वाहा समुद्राय स्वाहा,
  मध्याय स्वाहान्ताय स्वाहा . . . परार्धाय स्वाहा [4]

  Hail to hundred, . . . hail to hundred thousand . . . hail to hundred million . . . hail to trillion.

- We also find a list of odd numbers and multiples of four occuring in *Taittirīya-saṃhitā* (4.5.11):
  - एका च मे तिस्रश्च मे पञ्च च मे . . . एकत्रिꣳशच्च मे त्रयस्त्रिꣳशच्च मे
  - चतस्रश्च मेऽष्टौ च मे द्वादश च मे . . . चतुश्चत्वारिꣳशच्च मेऽष्टाचत्वारिꣳशच्च मे

[4] *Taittirīya-saṃhitā* 7.2.49.

Mathematical references in Vedas

Citations that unambiguously point to the decimal system being in vogue

- In the second *Maṇḍala* of *Ṛg-veda* we find multiples of ten listed.

  ता विंशत्या त्रिंशता या ह्यर्वाक् चत्वारिंशता हरिभिर्युजानः।
  ता पञ्चाशता सुरसेभिरिन्द्र षष्ट्या सप्तत्या सोम पेयम्॥
  अशीत्या नवत्या याह्यर्वाक् शतेन हरिभिरुह्यमानः।
  अयं हि ते ... [*Ṛg-veda* 2.18.5-6.]

  O Indra, Please come with twenty, thirty, forty horses ... with sixty, seventy ... carried by hundred horses.

- Elsewhere we find the appearance of an interesting number 3339.

  त्रीणि शता त्रीसहस्राण्यग्निं त्रिंशश्च देवा नव चासपर्यन्।
  औक्षन् घृतैरस्तृणन् बर्हिरस्मा ... [*Ṛg-veda* 3.9.9.]

  It is interesting to note that $3339 = 33 + 303 + 3003$, and is also close to 18 years ($\approx$ period of eclipse cycle).

- We also find a *mantra* referring to the notion of infinite (∞).

  पूर्णमदः पूर्णमिदं ...
  पूर्णस्य पूर्णमादाय पूर्णमेवाव शिष्यते॥

Mathematics in the *Śulbasūtra* texts

What are these texts, and where do they fall in the Vedic corpus?

- One of the prime occupations of the vedic people seem to have been performing sacrifices, for which altars of prescribed shapes and sizes were needed.
- Recognizing that manuals would be greatly helpful in constructing such altars, the vedic priests have composed a class of texts called *Śulba-sūtras*.
- These texts (earliest of which is dated prior to 800 BCE), form a part of much larger corpus known as *Kalpasūtras* that include:
  - श्रौत – Employed in rituals associated with societal welfare.
  - गृह्य – Rituals related to household.
  - धर्म – Duties[5] and General code of conduct.
  - शुल्ब – Geometry of the construction of fire-altar.

[5] *Ādi Śaṅkara* in his commentary on *Upaniṣads* defines the term *dharma* as *anuṣṭheyānāṃ sāmānyavacanam*.

What does the word *Śulba* mean?

- The word *śulba* stems from the root शुल्ब-माने (to measure).
- The etymological derivation of the word can be presented in more than one way:

भावव्युत्पत्ति – शुल्बनम् = शुल्बः।
Refers to the act of measuring.

कर्मव्युत्पत्ति – शुल्ब्यते इति शुल्बः।
Refers to the entity/result of measuring.

करणव्युत्पत्ति – शुल्बयत्यनेन इति शुल्बः।
Refers to the instrument of measuring.

- The complete derivation of the compound word *Śulbasūtras*, including the grammatical peculiarities is:

शुल्बनम् = शुल्बः (शुल्ब् + घञ्)[6]। तत्सम्बन्धि सूत्राणि।

[6]This type of derivation based on *bhāvavyutpatti* is governed by the *sūtra* '*bhāve ghañ*'.

The extant *Śulbasūtras*

- So far seven different *Śulbasūtra* texts have been identified by scholars. They are:
  1. *Baudhāyana Śulbasūtra*
  2. *Āpastamba Śulbasūtra*
  3. *Kātyāyana Śulbasūtra*
  4. *Mānava Śulbasūtra*,
  5. *Maitrāyaṇa Śulbasūtra*
  6. *Vāraha Śulbasūtra* and
  7. *Vādhūla Śulbasūtra*
- Of them, *Bodhāyana Śulbasūtra* is considered to be the most ancient one.[7] (prior to 800 BCE).
- It also presents a very systematic and detailed treatment of several topics that are skipped in later texts.
- It is made up of three chapters constituting about 520 *sūtras* $(113 + 83 + 323)$.

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[7] This assessment is based upon the style, completeness, and certain archaic usages that are not that frequently found in later texts.

Commentaries on *Śulbasūtras*

The table below presents a list of some of the important commentaries on three 'earlier' *Śulbasūtras*:

Śulbasūtra	Name of the comm.	Author
Bodhāyana	*Śulbadīpikā*	Dvārakānātha Yajvā
	Śulba-mīmāṃsā	Venkaṭeśvara Dīkṣita
Āpastamba	*Śulbavyākhyā*	Kapardisvāmin
	Śulbapradīpikā	Karavindasvāmin
	Śulbapradīpa	Sundararāja
	Śulbabhāṣya	Gopāla
Kātyāyana	*Śulbasūtravivṛtti*	Rāma/Rāmacandra
	Śulbasūtravivaraṇa	Mahīdhara
	Śulbasūtrabhāṣya	Karka

Qualities of a *Śulbakāra*

- Mahīdhara (c. 17th cent) in his *vivṛti* on *Kātyāyanaśulbasūtra* succinctly describes the qualities of a *śulbakāra*.

  सङ्ख्याज्ञः परिमाणज्ञः समसूत्रनिरञ्छकः।
  समसूत्रौ भवेद्विद्वान् शुल्बवित् परिपृच्छकः॥
  शास्त्रबुद्धिविभागज्ञः परशास्त्रकुतूहलः।
  शिल्पिभ्यः स्थपतिभ्यश्चाप्याददीत मतीः सदा॥
  तिर्यङ्मान्याश्च सर्वार्थः पार्श्वमान्याश्च योगवित्।
  करणीनां विभागज्ञः नित्योद्युक्तश्च सर्वदा॥

  A *śulbakāra* must be versed in arithmetic, versed in mensuration, . . . must be an inquirer, quite knowledgeable in one's own discipline, must be enthusiastic in learning other disciplines, always willing to learn from [practising] sculptors and architects . . . and one who is always industrious.

- The above anonymous citation clearly brings forth the point that a *śulbakāra*, is far more than a mere geometer.

Topics covered in the *Baudhāyana-śulbasūtra*

Sanskrit name	Their English equivalent
रेखामानपरिभाषा	Units of linear measurement
चतुरश्रकरणोपायः	Construction of squares, rectangles, etc.
करण्यानयनम्	Obtaining the surds/Theorem of the square of the diagonal
क्षेत्राकारपरिणामः	Transformation of geometrical figures
नानाविधवेदिविहरणम्[8]	Plan for different sacrifical grounds (*dārśa, paśubandha, sautrāmaṇi, agniṣṭoma* etc.)
अग्नीनां प्रमाणक्षेत्रमानम्	Areas of the sacrificial fires/altars
इष्टकसङ्ख्यापरिमाणादिकथनम्	Specifying the number of bricks used in the construction of altars including their sizes and shapes.
इष्टकोपधाने रीत्यादिनिर्णयः	Choosing clay, sand, etc. in making bricks
इष्टकोपधानप्रकारः	Process of manufacturing the bricks
श्येनचिदाद्याकारनिरूपणम्	Describing the shapes of *śyenaciti*, etc.

[8] In fact the text commences with the *sūtra*

अथेमेऽग्निचयाः । *(Now we describe the fire altars).*

Expression for the surds given in *Śulbasūtra* texts

- Besides presenting the details related to the construction of altars—that generally possess a bilateral symmetry—the *Śulba-sūtra* texts also present different interesting approximations for surds.
- The motivation for presenting estimates of surds could be traced to the attempts of vedic priests
  - to solve the problem of "squaring a circle" and vice versa
  - to construct a square whose area is n times the area of a give square, and so on.
- The expressions for surds presented in the form

$$N = N_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} + \dots,$$

can be understood in different ways, of which we will describe the Geometrical construction.

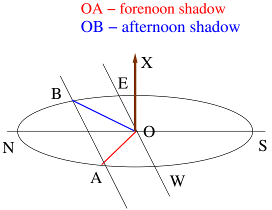
Topics that we plan to discuss

- Finding the cardinal directions using $\acute{s}a\dot{n}ku$.
- Construction of perpendicular bisectors.
- Construction of rectilinear (square, trapezia, etc.) and curvilinear (circles, *vedis*, etc.) geometrical objects.
- Enunciation of geometric principles and practical application of them. We demonstrate it with
  - Transformation of one geometrical object into another by applying these principles.
  - Obtaining the value of surds by means of geometrical construction.
- Estimating the value of surds (in the form of a sequence of rational numbers).
- Construction of altars (*citis*) of different sizes and shapes (falcons, tortoise, chariot wheel, and so on).

Dertermining the east-west line

- Determining the exact east-west line at a given location, is a pre-requisite for all constructions, be it a residence, a temple, a sacrificial altar or a fire-place.
- The procedure for its determination is described thus:

  समे शङ्कुं निखाय 'शङ्कुसम्मितया रज्ज्वा' मण्डलं परिलिख्य यत्र लेखयोः शङ्क्वग्रच्छाया निपतति तत्र शङ्कू निहन्ति, सा प्राची। [*Kt. Su.* I 2]



Fixing a pin (or gnomon) on levelled ground and drawing a circle with a cord measured by the gnomon,[9] he fixes pins at points on the line (of the circumference) where the shadow of the tip of the gnomon falls. That gives the east-west line (*prācī*).

[9] This prescription implies $r > 2OX$, and has astronomical significance.

Time from shadow measurement (not in *Śulbasūtras*)

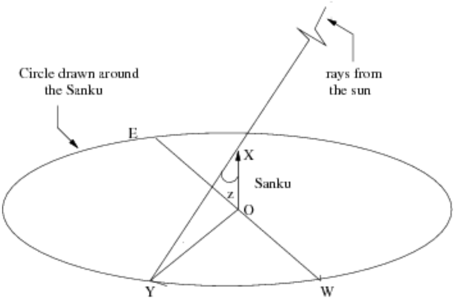


Figure : Zenith distance and the length of the shadow.

$$t = (R\sin)^{-1}\left[\frac{R\cos z}{\cos\phi\cos\delta} \pm R\sin\Delta\alpha\right] \mp \Delta\alpha.$$

If ϕ and δ are known ($\Delta\alpha = f(\phi, \delta)$), then t is known.

Why perform experiment to dertermine the directions?

- Posing the question – why not simply look at the sunrise or sunset, and be with it to find the east? – the commentator Mahīdhara observes:

  …तस्य उदयस्थानानां बहुत्वात् प्रतिदिनं भिन्नत्वात् अनियमेन प्राची ज्ञातुं न शक्या। तस्मात् शङ्कुस्थापनेन प्राचीसाधनमुक्तम्। दक्षिणायने चित्रापर्यन्तमर्कोऽभ्युदेति। मेषतुलासङ्क्रात्यहे प्राच्यां शुद्धायामुदेति। ततोऽर्कात् प्राचीज्ञानं दुर्घटम्।

  Since the rising points are many, varying from day to day, the [cardinal] east point cannot be known [from the sunrise point]. Therefore it has been prescribed that the east be determined by fixing a *śaṅku*. …Therefore, simply looking at the sun and determining the east is difficult.

- Having obtained the east-west direction, the next problem is to find out north-south. How to do that?

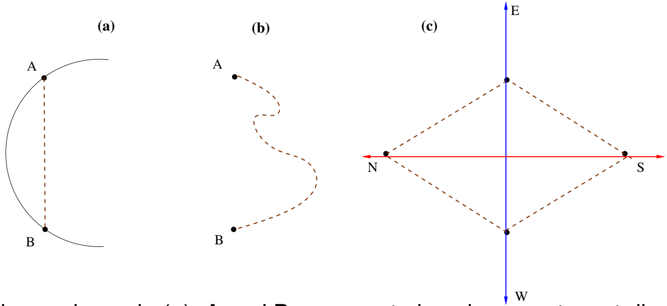
Methods for obtaining perpendicular bisector

- Two methods have been described for obtaining the perpendicular bisector of a given straight line:
  - रज्ज्वभ्यसनम् (folding the cord)
  - मत्स्यचित्रणम् (drawing fish-figure)
- How to draw a perpendicular bisector using the cord-folding method is discussed by Kātyāyana, in the third *sūtra* right at the beginning of his text.
- Having obtained *prācī*, getting *udicī* (the north-south line), correctly is extremely important for the construction of various altars having bilateral symmetry.

Construction of perpendicular bisector: Cord-folding method

तदन्तरं रज्वाभ्यस्य, पाशौ कृत्वा, शङ्कोः पाशौ प्रतिमुच्य, दक्षिणायम्य मध्ये शङ्कुं निहन्ति। एवमुत्तरतः, सोदीची।

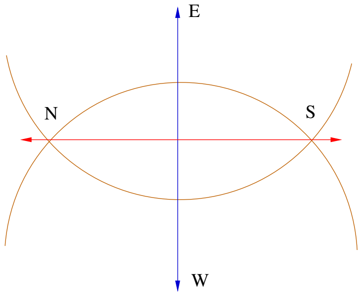
Doubling the cord by a measure of distance between them (*śaṅkus*), . . . and stretching (the cord) towards the south, strikes a pin at the middle point . . . .



In the figure above, in (a), A and B represent pins along east-west direction to which the cord is tied. In (b), we've doubled the cord AB. (c) represents stretching AB on both sides to get the north-south direction.

Construction of perpendicular bisector: Fish-figure method

- In this method, as shown in the figure below, having obtained the east-west direction by the shadow of the *śaṅku*, we mark two points along the east-west line.
- With those points as centres, and choosing an appropriate radius, circular arcs are drawn.
- The line passing through the intersection points of these two arc gives the north-south direction.

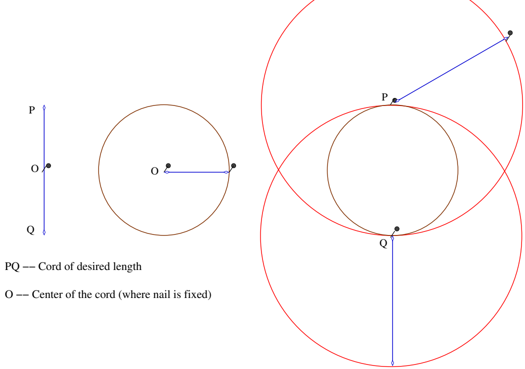


Bodhayana's method of constructing a square

Systematic procedure that involves cord & nails, but NO OTHER MEASURING DEVICE

चतुरश्रं चिकीर्षन् यावच्चिकीर्षेत् तावतीं रज्जुं उभयतः पाशं कृत्वा मध्ये लक्षणं करोति।
लेखामालिख्य तस्य मध्ये शङ्कुं निहन्यात्। तस्मिन् पाशौ प्रतिमुच्य लक्षणेन मण्डलं
परिलिखेत्। विष्कम्भान्तयोः शङ्कू निहन्यात्। पूर्वस्मिन् पाशं प्रतिमुच्य

Desirous of constructing a square, may you take a cord of that length, tie it at both the ends and mark its centre. Draw a line and fix a nail at its centre. Latching the ends . . .



Bodhayana’s method of constructing a square

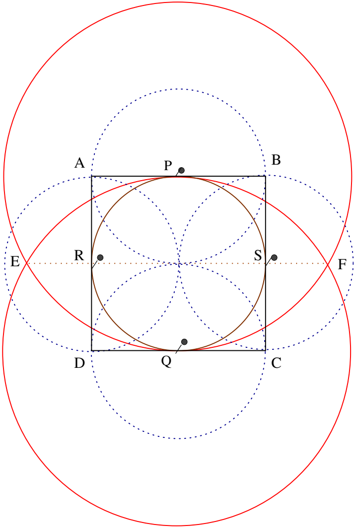
Systematic procedure that involves cord & nails, but NO OTHER MEASURING DEVICE

पाशेन मण्डलं परिलिखेत्। एवमपरस्मिन्। ते यत्र समेयातां, तेन द्वितीयं विष्कम्भं आयच्छेत्। विष्कम्भान्तयोः शङ्कू निहन्यात्। पूर्वस्मिन् पाशौ प्रतिमुच्य लक्षणेन मण्डलं परिलिखेत्। एवं दक्षिणेन एवं पश्चात्देवमुत्तरतः। तेषां ये अन्त्याः संसर्गाः तच्चतुरश्रं सम्पद्यते।

May you draw a circle.

Similarly on the other side.

From their points of instersection (E,F), obtain the second diameter (RS) . . .



The *Śulva* (Pythagorean?) theorem

- A clear enunciation of the so-called 'Pythagorean' theorem — called *bhujā-koṭi-karṇa-nyāya* in the later literature — is described in *Bodhāyana Śulvasūtra* (1.12) as follows:

  दीर्घचतुरश्रस्य अक्ष्णयारज्जुः[10] पार्श्वमानी तिर्यङ्मानी च यत् पृथग्भूते कुरुतः तदुभयं करोति।

  The rope corresponding to the the diagonal of a rectangle makes whatever is made by the lateral and the vertical sides individually.

Terms		their meaning
दीर्घचतुरश्रम्	–	Rectangle (lit. longish 4-sided figure)
अक्ष्णया रज्जुः	–	the diagonal rope
पार्श्वमानी	–	the measure of the lateral side
तिर्यङ्मानी	–	the measure of the perpendicular side

[10] The word *akṣṇayā* is archaic and hardly occurs in classical literature:

अक्ष्णया व्याघारयति। ... तस्मादक्ष्णया पशवोऽङ्गानि प्रतितिष्ठन्ति।

Kātyāyana version of *Śulva* theorem (with comm.)

- The Kātyāyana version of the theorem seems to be a redacted form of what appears in *Bodhāyana Śulvasūtra*.

  दीर्घचतुरश्रस्य अक्ष्णयारज्जुः तिर्यङ्मानी पार्श्वमानी च यत्
  पृथग्भूते कुरुतः तदुभयं करोति इति क्षेत्रज्ञानम्। *[KSS 2.7]*

- But for swapping two words, there is only one difference; The phrase '*iti kṣetrajñānam*' has been added ⇒ that this is the most fundamental theorem in geometry to be known whose knowledge cannot be dispensed with.
- Commenting on this *Mahīdhara* observes:

  दीर्घचतुरश्रस्य तिर्यङ्मानीपार्श्वमान्यौ रज्जू पृथग्भूते सत्यौ यत्क्षेत्रं
  = यत्फलकं क्षेत्रं समचतुरश्रद्वयं कुरुतः, तदुभयमपि मिलितं
  दीर्घचतुरश्रस्य अक्ष्णया = कोणसूत्रभूता रज्जुः करोतीति इति
  क्षेत्रज्ञानम् = क्षेत्रमानप्रकारो ज्ञातव्यः।

Mānava version of the *Śulva* theorem

- The presentation of the theorem in *Mānava-śulvasūtra* differs from *Bodhāyana Śulvasūtra* both in form and in style.
- Here it is given in the form of a verse as follows:

आयामं आयामगुणं विस्तारं विस्तरेण तु।
समस्य वर्गमूलं यत् तत् कर्णं तद्विदो विदुः॥

Terms		their meaning
आयामं आयामगुणं	–	the length multiplied by itself
विस्तारं विस्तरेण तु	–	and indeed the breadth by itself
समस्य वर्गमूलं	–	the square root of the sum
तत् कर्णम्	–	that is hypotenuse
तद्विदो विदुः	–	those versed in the discipline say so

- Using modern notation the result may be expressed as:

$$\sqrt{\bar{a}y\bar{a}ma^2 + vist\bar{a}ra^2} = kar\d{n}a.$$

Some ‘Pythagorean’ triplets listed in *Śulbasūtras*

- In the very next *sūtra* following the statement of the theorem, Bodhāyana illustrates it with a few examples:

तासां त्रिकचतुष्कयोः, द्वादशिकपञ्चिकयोः, पञ्चदशिकाष्टिकयोः,
सप्तिकचतुर्विंशिकयोः, द्वादशिकपञ्चत्रिंशिकयोः,
पञ्चदशिकषट्त्रिंशिकयोः इत्येतासु उपलब्धिः। *[BSS 1.13]*

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 5^2 + 12^2 &= 13^2 \\ 15^2 + 8^2 &= 17^2 \\ 7^2 + 24^2 &= 25^2 \\ 12^2 + 35^2 &= 37^2 \\ 15^2 + 36^2 &= 39^2 \end{aligned}$$

What is interesting to note is the use of the phrase

इत्येतासु उपलब्धिः।

[the general rule stated above] is quite evident in these pairs.

Is there a rationale behind the choice of these examples?

- A few triplets listed in the *Āśvalāyana-śulbasūtra* includes:

$$(15, 20, 25) \quad (16, 12, 20)$$

Rationale behind the choice of examples

Conjecture put forth by Datta (pp. 133-136)

- One of the Kātyāyana-sūtras presents the relation

$$na^2 = \left(\frac{n+1}{2}\right)^2 a^2 - \left(\frac{n-1}{2}\right)^2 a^2$$

- Substituting $n = m^2$, and $a = 1$, we at once get

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2 . \qquad (1)$$

- Here, putting $m = 3, 5, 7$ immediately $\rightsquigarrow$ (3,4,5), (5,12,13), (7,24,25).
- Rewriting the above equation in the form

$$(2m)^2 + (m^2-1)^2 = (m^2+1)^2, \qquad (2)$$

and substituting $m = 2, 4, 6 \rightsquigarrow$ (3,4,5), (8,15,17), (12,35,37).

- How about the other example of Bodhāyana (15,36,39)?

Principle behind generating right-rational triangles

Described by Āpastamba in the context *Saumikī-vedi*

त्रिकचतुष्कयोः पञ्चिका अक्ष्णयारज्जुः। ताभिः त्रिरभ्यस्ताभिः अंसौ। चतुरभ्यस्ताभिः श्रोणी। *[ASS 5.3]*

द्वादशिकपञ्चिकयोः त्रयोदशिका अक्ष्णयारज्जुः। ताभिः अंसौ। द्विरभ्यस्ताभिः श्रोणी। *[ASS 5.3]*

$$
\begin{array}{rcll}
3^2 + 4^2 & = & 5^2 & \\
(3 + 3.3)^2 + (4 + 3.4)^2 & = & (5 + 3.5)^2 & (A) \\
12^2 + 16^2 & = & 20^2 & \\
(3 + 4.3)^2 + (4 + 4.4)^2 & = & (5 + 4.5)^2 & (B) \\
15^2 + 20^2 & = & 25^2 & \\
5^2 + 12^2 & = & 13^2 & \\
(5 + 2.5)^2 + (12 + 2.12)^2 & = & (13 + 2.13)^2 & (C) \\
15^2 + 36^2 & = & 39^2 &
\end{array}
$$

It seems Āpastamba has invoked the principle that if (a, b, c) satisfies the relation $a^2 + b^2 = c^2$, then (ma, mb, mc) also satisfies the same relation—where m is an arbitary rational number.

Thanks!

THANK YOU

More about *Śulbasūtras* in the next lecture!