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BRAHMAGUPTA'S GENERAL SOLUTION OF 2ND ORDER INDETERMINATE EQUATION $Nx^2 + 1 = y^2$

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Abstract: In the present paper we have discussed the remarkable contribution of Brahmagupta (628 AD) in providing the general solution of the 2nd order indeterminate equation of the *varga prakiti* type $Nx^2 + 1 = y^2$ (wrongly called Pell's Equation).any arbitrary positive integer; coefficient of x^2 is N; difference; multiplied by two; x; square; added to one (when in the thence; y; infinity; (the number of solutions will be infinite) by the cyclic method; so also; by choosing the arbitrary number].

The three lemmas of Brahmagupta lead to his method of *samsa bhvan* and *antara bhvan*, in obtaining the general solution. His method is illustrated with examples. Further, a geometrical interpretation of the *varga prakti* equation is also provided. Recursion formula for the *tulya bhvan* and the *samsa bhvan* are provided in the paper to enable writing a computer program.

1. Introduction

In the theory on numbers and algebra sustained investigations have been made for several centuries. In ancient Greece, Diophantus is credited with studying indeterminate equations, and hence such equations are named after him. However Diophantus did not provide methods of general solutions of these equations. In fact the credit of providing the methods of general solutions of the 1st degree indeterminate equations of the form $ax - by = c$ and of the 2nd degree of the form $Nx^2 + 1 = y^2$ comes exclusively to ancient India. While ryabhata I (b. 476 AD) has given the method of solution of $ax - by = c$ type (called *kuaka*), the 2nd order equations of the type $Nx^2 + 1 = y^2$ (*varga prakti*) was completely solved by Brahmagupta (628 AD). Brahmagupta discusses the topic in detail in the18th century of his magnum opus, *Brahmasphuta Siddhnta*. After Brahmagupta, Indian mathematicians like Mahvira (9th century) the Jain mathematician of Karnataka, Jayadeva (10–11th century) and Bhskar II (b. 1114) improved the techniques. In Brahmagupta's method we will have to start with a guessed value for c, being the difference between the given constant non-square integer N and its closest square. But in the Cakravla methods of Jayadeva and Bhskar II even this need to guess the initial value is dispensed. The algorithm provided by these remarkable mathematicians or easily amenable to computer programming. In fact it has been proved both theoretically (A. Krishnaswamy Iyengar) and also on the computer that the algorithm of Bhskar's procedure is simpler and faster than those of Euler (1764) and Lagrange (1768).

2. Brahmagupta's Lemmas

Lemma 1: If $(x, y) = (á, \beta)$ is a solution of the equation

$$Nx^2 + c = y^2 \qquad (1)$$

and $(x, y) = (á', \beta')$ is a solution of the equation

$$Nx^2 + c' = y^2 \qquad (2)$$

then

$$(x, y) = \begin{matrix} (á\beta' + á'\beta,\ \beta\beta' + Ná á') \\ (á\beta' - á'\beta,\ \beta\beta' - Ná á') \end{matrix}$$

are solutions of the equation

$$Nx^2 + cc' = y^2.$$

Proof: We have

$$Nx^2 + c = y^2 \text{ and } Nx^2 + c' = y^2$$

Taking

$$(x, y) = (á\beta' + á'\beta,\ \beta\beta' + Ná á')$$

we see that

$$\begin{aligned} y^2 - Nx^2 &= (\beta\beta' + N\alpha\alpha')^2 - N(\alpha\beta' + \alpha'\beta)^2 \\ &= (\beta^2\beta'^2 + N^2\alpha^2\alpha'^2 + 2N\alpha\alpha'\beta\beta') - N(\alpha^2\beta'^2 + \alpha'^2\beta^2 + 2\alpha\alpha'\beta\beta') \\ &= \beta'^2(\beta^2 - N\alpha^2) - N\alpha'^2(\beta^2 - N\alpha^2) \\ &= (\beta^2 - N\alpha^2)(\beta'^2 - N\alpha'^2) \\ &= cc'. \end{aligned}$$

Similarly it can be verified that

$$(x, y) = (\alpha\beta' - \alpha'\beta,\ \beta\beta' - Ná á')$$

satisfies the equation

$$Nx^2 + cc' = y^2$$

Lemma 2: If $(á, \beta)$ is a solution of the equation

$$Nx^2 + c^2 = y^2$$

Then $(x, y) = (2\alpha\beta, \beta^2 + N\alpha^2)$ is a solution of the equation

$$Nx^2 + c^2 = y^2.$$

Proof: This follows from Lemma 1 when we take $c' = c$ and $(\alpha', \beta') = (\alpha, \beta)$.

It can also be directly verified as follows:

$$\begin{aligned} y^2 - Nx^2 &= (\beta^2 + N\alpha^2)^2 - N(2\alpha\beta)^2 \\ &= (\beta^2 + N\alpha)^2 - 4\beta^2N\alpha^2 \\ &= (\beta^2 - N\alpha^2)^2 = c^2 \end{aligned}$$

since $N\alpha^2 + c^2 = \beta^2$.

Lemma 3: If $(x, y) = (\alpha, \beta)$ is a solution of the equation $Nx^2 + c^2 = y^2$ such that α/c and β/c are integers then $(\alpha/c, \beta/c)$ is a solution of the equation $Nx^2 + 1 = y^2$.

Proof: We have that $\alpha' = \alpha/c$ and $\beta' = \beta/c$ are integers, and

$$\beta'^2 - N\alpha'^2 = (\beta^2/c^2) - N(\alpha^2/c^2) = (1/c^2)(\beta^2 - N\alpha^2) = 1$$

since $Nx^2 + c^2 = y^2$.

Remark 1: If we take $c = 1$ in Lemma 2 we see that, if (α, β) is a solution of $Nx^2 + 1 = y^2$, then $(2\alpha\beta, \beta^2 + N\alpha^2)$ is a solution of $Nx^2 + 1 = y^2$.

Remark 2: We shall see later that the solution of the equation $Nx^2 + 1 = y^2$ is always possible. However the equation $Nx^2 - 1 = y^2$ may not have integer solution.

3. Samsa Bhvan

When N and c are integers in the Brahmagupta equation $Nx^2 + c = y^2$, to get the integral values also for x, y the *vargaprakiti* has come into existence. Since (*varga*) square of x is multiplied by the (*prakiti*) coefficient N, the name *vargaprakti* is justified.

Prof. K. S. Sanjana translated the word *vargaprakriti* to English as "the multiplied square" and Prof. Colebrooke as "the method of affected square" and Datta and Singh as "the method of the square nature".

The term *vargaprakiti* has been used by our ancient Indian mathematicians to designate an equation of the form $Nx^2 + c = y^2$ where N and c are given positive integers. N the coefficient of x^2, is called *praiti*, x is called *kaniha pada, hrasva pada, adyamla* or the lesser root; y is called *jyeha pada, anymla* or the greater root; $\pm$ c is called the *kepa, kepaka, prakepa* or the interpolator. The most fundamental equation of this type has been regarded as $N^2 + c = y^2$.

No ancient Indian mathematician discusses the values of c by giving the universal value to it. But utmost importance is given to the equation where $c = 1$. This value is illustrated here in brief.

Conveniently choosing the values for c, c', consider

$$N\alpha^2 + c = \beta^2$$

$$N\alpha'^2 + c' = \beta'^2$$

where one can choose integral values for α, β and α', β'. Then $\alpha\beta' + \alpha'\beta$ and $\beta\beta' + N\alpha\alpha'$ are the solutions of $Nx^2 + cc' = y^2$ (Lemma 1). The same result was being rediscovered by Euler (1764) and Lagrange (1768).

The above method of Brahmagupta of getting the integral values for x and y is called *samasa bhvan* (composition method). These results may be conveniently expressed in tabular form as follows:

1. *samyukta bhvan* (additive composition)

prakithi	**kanitha**	**jyetha**	**kepa**
N	α	β	c
	α'	β'	c'
	$\alpha\beta' + \alpha'\beta$	$\beta\beta' + N\alpha\alpha'$	cc'

2. *antara bhvan* (subtraction composition)

prakithi	kanitha	jyetha	kepa
N	α	β	c
	α'	β'	c'
	$\alpha\beta' - \alpha'\beta$	$\beta\beta' - N\alpha\alpha'$	cc'

3. *tulya bhvan* (composition of equals)

prakithi	kanitha	jyetha	kepa
N	α	β	c
	α	β	c
	$2\alpha\beta$	$\beta^2 + N\alpha^2$	c^2

$(\alpha\beta' + \alpha'\beta, \beta\beta' + N\alpha\alpha', cc')$, $(\alpha\beta' - \alpha'\beta, \beta\beta' - N\alpha\alpha', cc')$, $(2\alpha\beta, \beta^2 + N\alpha^2, c^2)$ are the three composed values and can be remembered very well.

If further α, β, c are composed with again α, β, c, we get

$N(2\alpha\beta)^2 + c^2 = (\beta^2 + N\alpha^2)^2$.

Therefore,

$$N\left(\frac{2\alpha\beta}{c}\right)^2 + 1 = \left(\frac{\beta^2 + N\alpha^2}{c}\right)^2.$$

Here $x = 2\alpha\beta/c$, $y = \beta^2 + N\alpha^2/c$ are the solution of $Nx^2 + 1 = y^2$. The part of the work is not completed unless the values are integers.

In the Brahmagupta equation $N\alpha^2 + c = \beta^2$, the values which are so chosen that $c = \pm 1$, or $c = \pm 2$, the above said values x, y becomes integral values but if c = 4 then $(\alpha/2, \beta/2, 1)$ can be composed itself to get, $x = ½\alpha\beta$, $y = ½(\beta^2 - 2)$ again composing $(\alpha/2, \beta/2, 1)$ and $½\alpha\beta$, $½(\beta^2 - 2)$, 1 we get, $x = ½\alpha(\beta^2 - 1)$, $y = ½\beta(\beta^2 - 3)$ are the solutions obtained by Brahmagupta. He says if the values of α in even (hence the value of β also), then β though it gives integral values. If β is even, it is sufficient to continue with previous values. Likewise, if c = –4 then,

$x = ½\alpha\beta(\beta^2 + 1)(\beta^2 + 3)$

$y = (\beta^2 + 2)\{½(\beta^2 + 1)(\beta^2 + 3) - 1\}$

are the two solutions obtained by Brahmagupta. Hence by choosing conveniently the values $c = \pm 1$, $c = \pm 2$, $c = \pm 4$ according to the situation one can get the solution for $Nx^2 + 1 = y^2$. After getting a pair of solution by composing them repeatedly with the same pair of solution, infinitely many numbers of solutions can be obtained.

There ends Brahmagupta method. Values of α, β are to be determined from the equation $N\alpha^2 + c = \beta^2$ by choosing suitable values for $c = \pm 1, \pm 2$ and ± 4. Brahmagupta himself did not show the method of finding them. Bhaskara by *cakravla* (cyclic Method after Brahmagupta has completed the remaining part of the work. In western countries at the time of 628 AD it was so difficult even the calculation of simple multiplication and division, Brahmagupta's this high level mathematics was very difficult to understand and digest and also to accept.

4. Illustrations

which square; multiplied by eight; added to one; square; mathematician; multiplied by 11; or; O friend].

Ex: (1) $8x^2 + 1 = y^2$ (2) $11x^2 + 1 = y^2$

Consider example, (1) $8x^2 + 1 = y^2$

Solution: By trial, $x = 1$, $y = 3$. Now use *tulya bhvan*,

N	x	y	c
8	1	3	1
	1	3	1
	$2\alpha\beta$	$\beta^2 + N\alpha^2$	c^2
	$2 \times 1 \times 3$	$3^2 + 8 \times 1^2$	$1^2 = 1$
	= 6		= 17

$\therefore x = 5, y = 17$, ………………….second set

Again combining the above two solutions by additive *bhvan* we get,

N	x	y	c
8	6	17	1
	1	3	1
	$6 \times 3 + 1 \times 17$	$17 \times 3 + 8 \times 6 \times 1$	$1^2 = 1$
	= 35	= 99	= 1

$\therefore x = 35, y = 99$ ………………….third set.

2. $11x^2 + 1 = y^2$

Solution: By trial, x=1, y=3. Then, by *tulya bhavana*, we obtain x=60, y=199 as the second solution.

$\therefore 31 \times (21)^2 + 18 = 117^2$.

3. $14 \times x^2 + 1 = y^2$.

Solution: By trial, $x = 1$, $y = 4$. Now, use *tulya bhvan*,

N	x	y	c
14	1	4	2
	1	4	2
	$2 \times 1 \times 4$	$4^2 + 14 \times 1^2$	$2^2 = 4$
	= 8	= 30	= 4

$14 \times 8^2 + 4 = 30^2$.

Dividing by 4, we get $14 \times 4^2 + 4 = 15^2$. Giving the solution $x = 4$, $y = 15$ of (1).

4. $31x^2 + 1 = y^2$.

Solution: Here it is not easy to guess an auxiliary equation. Therefore, to obtain it, we first solve related equation corresponding to suitable values of *kepa*. Thus by trial, $31x^2 - 6 = y^2$ has solution $x = 1$, $y = 5$ and $31x^2 - 3 = y^2$ has solution $x = 2$, $y = 11$.

Now applying additive *bhavn* we get,

N	x	y	c
31	1	5	−6
	2	11	−3
	$1 \times 11 + 2 \times 5$	$5 \times 11 + 31 \times 1 \times 2$	$(-6) \times (-3)$
	$= 21$	$= 117$	$= 18$

$$\therefore\ 31 \times (21)^2 + 18 = 117^2.$$

Dividing by 3^2 we get, $31 \times 7^2 + 2 = 39^2$. $\therefore x = 7, y = 39$ is a solution of the auxiliary equation $31x^2 + 2 = y^2$.

Hence, by *tulya bhvan*,

N	x	y	c
31	7	39	2
	7	39	2
	$2 \times 7 \times 39$	$39^2 + 31 \times 7$	2^2
	$= 546$	$= 3040$	$= 4$

$$\therefore\ 31 \times (546)^2 + 4 = 3040^2.$$

Dividing by 4 we get $31 \times (273)^2 + 1 = 1520^2$.

$$\therefore\ x = 273,\ y = 1520.$$

5. References

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